Worksheet
Harmonic and Fibonacci

Name___________________
Period___Date___________

A **harmonic sequence** is a sequence in which all of the terms are reciprocals of the terms of an arithmetic sequence.

Ex. #1. Since 2, 4, 6, 8, … is arithmetic, 1/2, 1/4, 1/6, 1/8, is harmonic.
Ex. #2. Since $a_n = 4n - 7$ is arithmetic, $h_n = 1/(4n - 7)$ is harmonic.

To insert k harmonic means between numbers a and b, insert k arithmetic means between 1/a and 1/b, then write the reciprocals of those means.

Ex. ***Insert 2 harmonic means between 3 and 10.***

Consider the arithmetic sequence: 1/3, _____, _____, 1/10

Their common difference is (1/10 – 1/3)/(4 – 1) = –7/90, so the arithmetic sequence is:

1/3, 23/90, 16/90, 1/10 and that means the harmonic means are 90/23 and 90/16.

Exercises:

1. Insert 3 harmonic means between 1/3 and 1/9.
If the harmonic sequence is 1/3, __, __, ___, 1/9, then the arithmetic sequence is 3, _____, ____, _____, 9. The common difference for this arithmetic sequence is (9 – 3)/(5 – 1) = 6/4 = 3/2, so the arithmetic sequence is 3, 9/2, 6, 15/2, 9. A sequence of the reciprocals of these terms would be 1/3, 2/9, 1/6, 2/15, 1/9 which is harmonic. So the harmonic means are ***2/9, 1/6, 2/15***

2. Insert 5 harmonic means between 1/6 and 1/12.
We will insert 5 arithmetic means between 6 and 12. 6, __, __, __,__,__, 12. The common difference (slope) is (12 – 6)/(7-1) = 6/6 = 1. So the arithmetic sequence is 6, 7, 8, 9, 10, 11, 12 and the harmonic sequence is 1/6, 1/7, 1/8, 1/9, 1/10, 1/11, 1/12 so the means are ***1/7, 1/8, 1/9, 1/10, 1/11.***

3. Insert 2 harmonic means between 3 and 7.
Insert 2 arithmetic means between 1/3 and 1/7. 1/3, __, __, 1/7. The common difference (slope) is (1/7 – 1/3)/(4-1) = (-4/21)/3 = -4/63. So the arithmetic sequence is 1/3, 17/63, 13/63, 1/7 and the harmonic sequence is 3, 63/17, 63/13, 7 so the means are ***63/17, 63/13.***

4. Insert 3 geometric means between 2 and 9. *If 2, ____, ____, 9 form a geometric sequence,* $9 = 2r^3$, *so* $r = \sqrt[3]{9/2}$. *Means will be* $2\sqrt[3]{9/2}$, *and* $2\left(\sqrt[3]{9/2}\right)^2$.

5. Prove that {2, 3, 6} are terms of a harmonic sequence.
If 2, 3, and 6 form a harmonic sequence, then 1/2, 1/3, 1/6 should be arithmetic. This means 1/3 – 1/2 should equal 1/6 – 1/3. Well, 1/3 – 1/2 =-1/6 and 1/6 – 1/3 = -1/6, so they are equal Therefore 2, 3, 6 are harmonic. (Another way to show this is to show that {2, 3, 6} can also be written as $\left\{\frac{6}{3},\frac{6}{2},\frac{6}{1}\right\}$*, and this can be written as 6/(4-n) which is the reciprocal of an arithmetic sequence (4-n)/6.*

6. Given the numbers, 4 and 8:

A) Insert an arithmetic mean between the two. *6*

B) Insert a geometric mean between the two. $4\sqrt{2}$

C) Insert a harmonic mean between the two. *16/3*

D) Write these in order from lowest to highest. *16/3,* $4\sqrt{2}$ *, 6*

7. Given the positive numbers, a and b, write the arithmetic mean, the geometric mean and the harmonic mean and decide if the order (from lowest to highest) sill applies and why it does.

Arithmetic mean = $\frac{a+b}{2}$*, Geometric mean =* $\sqrt{a \cdot b}$ *Harmonic mean =* $\frac{2ab}{a+b}$

The order still works…

The Fibonacci sequence is the sequence of numbers, 1, 1, 2, 3, 5, 8, 13, 21, …

What is the next term of the sequence? 34

Write a recursive formula for this sequence. $f_n = f_{n-1} + f_{n-2}$

1. Take any three sequential Fibonacci numbers. Multiply the outer two and square the middle one. Try this activity with 3 different sequential Fibonacci numbers. What do you notice? *There is always a difference of 1.*

2. Take any four sequential Fibonacci numbers. Multiply the outer two and multiply the inner two. Try this again with 4 different sequential Fibonacci numbers. What do you notice? *The difference is always 1.*

3. Create a new sequence by finding f_n/f_{n+1}. (I use lists…) Does this sequence converge or diverge? If it converges, what number appears to be the limit?

It converges to $\varphi = \frac{1+\sqrt{5}}{2} = 1.6180339887\ldots.$

4. If n is a prime number, what appears to be true of f_n? *f_n is prime.*

5. Find two consecutive Fibonacci numbers, f_n and f_{n+1}. Then locate f_{2n+1} . What do you notice about these numbers? *$f_{2n+1} = f_n^2 + f_{n+1}^2$.*

6. Classify the following sequences as arithmetic, geometric, harmonic, Fibonacci, or other. Then write the n^{th} term (explicitly if possible).

A) 1, i, –1, -i, … *Geometric, $t_n = i^{n-1}$*

B) log 2, log 4, log 8, log 16, … *Arithmetic, t_n = 2log 2 (or log 2^n)*

C) 4/3, 4/5, 4/7, 4/9, … *Harmonic, t_n = 4/(2n+1)*

D) 1, 1, 2, 3, 5, 8, … *Fibonacci, $t_n = t_{n-1} + t_{n-2}$*

E) 12, 6, 4, 3, … *Harmonic, t_n = 12/n*

F) 3 – i, 3, 3 + i, 3 + 2i, … *Arithmetic, $t_n = 3 - 2i + n \cdot i$*

G) sin 0, sin π, sin 2π, sin 3π, sin 4π, … *Arithmetic (Constant) $t_n = 0$*

H) cos 0, cos π, cos 2π, cos 3π, cos 4π, … *Geometric, $t_n = (-1)^{n-1}$*

I) sin 0, sin 90°, sin 180°, sin 270°, sin 360°, … *Other $t_n = \sin(90° \cdot n)$*