Accelerated Precalculus

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4-10-18
3RD

Sequences and Series Study Guide

Unit 11

I) Arithmetic and Geometric Sequences

→ **arithmetic sequence** - a sequence with a common difference

- **Recursive formula** - tells you the next term given that you know the previous term(s)

$$\boxed{a_n = a_{n-1} + d}$$

a_n = nth term
a_{n-1} = previous term
d = common difference

- **explicit formula** - used to find any term of the sequence

$$\boxed{a_n = a_1 + (n-1)d}$$

a_n = nth term
a_1 = 1st term
n = # of terms
d = common difference

→ **geometric sequence** - a sequence with a common ratio

- **Recursive formula**

$$\boxed{a_n = a_{n-1} \cdot r}$$

a_n = nth term
a_{n-1} = previous term
r = common ratio

- **explicit formula**

$$\boxed{a_n = a_1 \cdot r^{n-1}}$$

a_n = nth term
a_1 = 1st term
n = # of terms
r = common ratio

II.) Arithmetic and geometric Series

→ **series**: the sum of the terms in a sequence

→ **sigma notation**

$$\sum_{n=\square}^{\square} \boxed{}$$

← last value of n
← formula for the sequence
← first value of n
index of summation ↑

→ arithmetic sums

$$\boxed{S_a = \frac{n}{2}(a_1 + a_n)}$$

- n = # of terms
- a_1 = first term
- a_n = nth term

* note: any fraction to the ∞ power = 0

→ geometric sums

$$\boxed{S_g = \frac{a_1(r^n - 1)}{r - 1}}$$

- a_1 = first term
- n = number of terms
- r = common ratio

III. Successive Differences and Polynomial Sequences

$$f(x) = ax^2 + bx + c$$

	n = 1		n = 2		n = 3	
f(x)	$\boxed{a+b+c}$		$4a+2b+c$		$9a+3b+c$	...
1st successive difference		$\boxed{3a+b}$		$5a+b$		$7a+b$
2nd successive difference			$\boxed{2a}$		$2a$	
3rd successive difference				0		0

→ to write equations...

Method 1
evaluate successive differences

$2a = \boxed{}$
• solve for a ⇒

$3a + b = \boxed{}$
• substitute in a
• solve for b ⇒

$a + b + c = \boxed{}$
• substitute in a and b
• solve for c

⇒ plug in a, b, and c into $f(x) = \underline{a}x^2 + \underline{b}x + \underline{c}$

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method 2
solve using systems of equations

given $f(x) = n_1, n_2, n_3$
and $f(x) = a+b+c,\ 4a+2b+c,\ 9a+3b+c$

use the system / matrix

$$\text{rref}\begin{bmatrix} 1 & 1 & 1 & n_1 \\ 4 & 2 & 1 & n_2 \\ 9 & 3 & 1 & n_3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{bmatrix}$$

where $f(x) = \underline{a}x^2 + \underline{b}x + c$

* the degree of the polynomial is equal to the number at which the differences are constant
ex) if third differences are constant, it should be cubic

(IV) Figurate Number Sequences

- **figurate numbers** - #s that can be represented geometrically by an arrangement of equally spaced points

- **figurate number sequence**:

  created by making larger and larger regular figures beginning with the smallest possible ($n=1$)

↳ solve equations for these by determining the sequence of # of dots in each figure, then use polynomial sequences / successive differences to solve for a, b, and c, then write formula

(V) Arithmetic and Geometric Means

→ arithmetic means

- 1 mean between 2 numbers

  $\underline{a}, \boxed{?}, \underline{b}$ • simply average the two $\boxed{\left(\frac{a+b}{2}\right)}$

- multiple means between 2 numbers

  $\underset{a}{\underline{a}}, \underset{a+d}{\underline{?}}, \underset{a+2d}{\underline{?}}, \underset{\ldots}{\underline{\ldots}}, \underline{b}$

  to find d: $\boxed{\dfrac{(b-a)}{(n-1)}}$

  [a = 1st term
  b = last term
  n = total # of terms

→ geometric means

- 1 mean between 2 numbers

  $\underline{a}, \underline{?}, \underline{b}$ $\quad ? = \boxed{\sqrt{ab}}$

- multiple means between 2 numbers

  $\underset{a}{\underline{a}}, \underset{ar}{\underline{?}}, \underset{ar^2\ldots}{\underline{?}} \underset{ar^{(n-1)}}{\underline{b}}$

to find r: $ar^{(n-1)} = b \Rightarrow r^{(n-1)} = \frac{b}{a} \Rightarrow \boxed{r = \sqrt[(n-1)]{\frac{b}{a}}}$

[n = nth term
r = ratio
a = 1st term
b = last term

VI Harmonic Sequences

→ harmonic sequence - sequence in which all of the terms are reciprocals of the terms of an arithmetic sequence

- graphs rational (inverse) functions

* try to create common NUMERATORS

→ harmonic means

① • to insert $\boxed{k}$ harmonic means between a and b insert $\boxed{k}$ aritmetic means between $\frac{1}{a}$ and $\frac{1}{b}$ and write the reciprocal of those means

② • write with common numerators and find arithmetic means of the denominators

VII Fibonacci Sequences

→ fibonacci sequence - each number is the sum of the two preceeding numbers

- recursive formula $\boxed{f_n = F_{n-1} + F_{n-2}}$

→ patterns

① f_n , f_{n+1}, f_{n+2}

$$(f_n * f_{n+2}) = 1 + (f_{n+1})^2$$

② f_n, f_{n+1}, f_{n+2}, f_{n+3}

$$(f_n * f_{n+3}) + 1 = (f_{n+1} * f_{n+2})$$

③ if n is prime, f_n appears to be prime

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XIII Special Sequences

→ nestled radicals

$$\sqrt{a+\boxed{\sqrt{a+\sqrt{a+\sqrt{a\ldots}}}}}$$

- set equal to x
- set the part that begins to repeat equal to x
- Solve for x using quadratics

→ nestled fractions

$$a+\cfrac{b}{\boxed{a+\cfrac{b}{a+\cfrac{b}{a+\cfrac{b}{\ldots}}}}}$$

- set equal to x
- set the part that begins to repeat equal to x
- Solve for x using quadratics

→ decomposition of fractions into components

$$\frac{\boxed{z}}{\boxed{y}}$$

- factor the denominator
- split the fraction into components

$$\frac{A}{\boxed{c}}+\frac{B}{\boxed{d}}$$

- create a common denominator

$$\frac{A\boxed{d}}{\boxed{c}\boxed{d}}+\frac{B\boxed{c}}{\boxed{c}\boxed{d}} \quad \text{where} \quad \boxed{c}\boxed{d}=\boxed{y}$$

- set equal to original

$$\boxed{z}=A\boxed{d}+B\boxed{y}$$

method 1: solve using systems of equations

→ gather like terms

method 2: solve using intercepts

- create values for x that will make $Ax=0$ and solve for B and $Bx=0$ and solve for A

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IX Power Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \ldots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} \ldots$$

or $\cos(x) = \frac{1}{2}(e^{ix} + e^{-ix})$

$$\sin(x) = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}$$

or $\sin(x) = \frac{1}{2i}(e^{ix} - e^{-ix})$

Exponential form of a complex #

$$\boxed{\cos\theta + i\sin\theta = e^{i\theta}}$$ or $\text{cis}(\theta) = e^{i\theta}$

$r\,\text{cis}\theta = re^{i\theta}$

hyperbolic cosine

$\cosh(x) = .5(e^x + e^{-x})$

hyperbolic sine

$\sinh(x) = .5(e^x - e^{-x})$

Remember
- $r = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1}(b/a)$ in $a + bi$ form
- $0! = 1$
- $i^2 = -1$ and $i = \sqrt{-1}$

X Telescoping Series

approach 1: write out each term until you can decide pattern & cancel inside terms to find sums

approach 2: decompose into partial fractions, then write out terms & look for patterns, cancel terms, find sum

XI Golden Ratio

Golden Rectangle: if you cut a square from the original rectangle, the remaining rectangle is similar to the original rectangle

Golden Ratio (phi) $\Phi \approx 1.618033989...$

$$\frac{1+\sqrt{5}}{2}$$

Golden Triangle: isoscoles triangle with leg and base lengths that are in the ratio Φ

→ fibonacci

$t_n = \frac{F_{n+1}}{F_n}$ where F_n is the n^{th} term of the Fibonacci sequence

↳ this sequence appears to approach Φ

Explicit Formula of Fibonacci sequence:

$$\boxed{F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}}$$

XII Principle of mathematical Induction

→ given a sequence, and an equation, prove it.

1. prove the first case for $n = 1$
   ↳ anchor the proof
2. let this be true for some $K \in N$
   ↳ assume this statement is true for some value of K
   - Replace variable with K
3. prove the case after the K^{th} case is also true
   - Rewrite portion obtained in part 2, but add the $(K+1)$ term to the sequence AND equation side
   - use algebra to substitute, factor, & solve to make both sides equal
4. "Since the Kth term implies the $(K+1)^{st}$ term, this proof holds true for all cases of the natural #s $(K \in N)$"
5. Therefore (original sequence/equation) by the Principle of Mathematical Induction