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3RD

Acc. Pre Calculus
Systems and matrices study guide
unit 1

I. Systems of Equations

methods of solving:

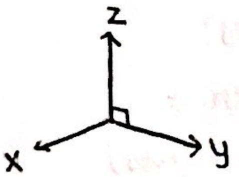
1. **substitution** isolate one variable and substitute it into the other equation(s), then solve for the variables
2. **elimination** choose one variable to eliminate
   - multiply so that coefficients are the same on one variable
     ↳ one must be negative, one must be positive
   - then add/subtract remaining equation and find variables
3. **graphing** graph each equation ($y=mx+b$) and find point(s) of intersection

→ If equations are the same or one is the inverse of the other (multiply by -1 to get same equation) ⇒ then answer is: infinitely many solutions

→ 2×2 systems vs 3×3 systems
- 2 equations — 3 equations
- 2 unknowns — 3 unknowns

* when solving a 3×3 and there is one equation w/ only 2 unknowns, eliminate the unknown that is distinct to the other two

II. Graphing in 3D



- any equation of the form: $Ax+By+Cz=D$ will graph a plane in 3D space

III. Matrices

1. **basics**
   - matrix: rectangular array of numbers
     ↳ dimensions: rows x columns
   - scalar: any real number that a matrix can be multiplied by
     ↳ can be multiplied by a matrix ⇒ ALWAYS
2. **addition**
   - can only be added if dimensions are the SAME
   - add each number with the number in the same position in the other matri

3. Multiplication

- can only be done when the number of COLUMNS in the first matrix equals the number of ROWS in the second matrix

yes: 1×3

no: 3×2 by 1×3

- How to multiply:
  1. multiply the top row with the left most column and add the products ⇒ place answer in top left corner
  2. multiply the top row with the second to left most column, add products + place answer to right of 1st answer
  3. repeat with top row until all columns in matrix 2 have been used
  4. use the next ROW of the first matrix and repeat by multiplying by the columns going left to right
  5. continue until all rows in the first matrix have been multiplied by all the columns in the second matrix

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} * \begin{bmatrix} j & k & l \\ m & n & o \\ p & q & r \end{bmatrix} = \begin{bmatrix} aj+bm+cp & ak+bn+cq & al+bo+cr \\ dj+em+fp & dk+en+fq & dl+eo+fr \\ gj+hm+ip & gk+hn+iq & gl+ho+ir \end{bmatrix}$$

4. Properties of matrices

a. **additive identity** - when you add something and it won't change the value

↳ a matrix of the same dimension filled with zeroes

b. **additive inverse** - when added with matrix, will result in additive identity (a matrix of zeroes)

↳ -1 [matrix]

c. **multiplicative identity** - the matrix that can be multiplied by another matrix that won't change the matrix's value

⇒ must be square (so dimensions are equal)

⇒ looks like a matrix filled with zeroes except for 1's forming a diagonal line ↘

d. **multiplicative inverse** - will result in the multiplicative identity when multiplied by the matrix

$[\text{matrix}]^{-1}$

5. The Determinant

- a value associated with any <u>SQUARE</u> matrix
- written as $\det\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ <u>OR</u> $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ → Note: these are <u>not</u> absolute value bars... determinants can be negative

→ 2x2

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

→ 3x3

$$\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a\begin{bmatrix} e & f \\ h & i \end{bmatrix} - b\begin{bmatrix} d & f \\ g & i \end{bmatrix} + c\begin{bmatrix} d & e \\ g & h \end{bmatrix} \Rightarrow$$

$a(ei-fh) - b(di-fg) + c(dh-eg)$

$\Downarrow$

$aei - afh - bdi + bfg + cdh - ceg$

→ expanding by minors
- can expand along any row or column
- make sure to use "checker board" pattern of $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$ when expanding

6. Inverses

→ 2 by 2

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

$ad-bc$ = determinant

→ 3 by 3

$$A = \left[\begin{array}{ccc|ccc} a & b & c & 1 & 0 & 0 \\ d & e & f & 0 & 1 & 0 \\ g & h & i & 0 & 0 & 1 \end{array}\right] \Rightarrow B = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & j & k & l \\ 0 & 1 & 0 & m & n & o \\ 0 & 0 & 1 & p & q & r \end{array}\right]$$

use gauss Jordan to change it from ~~A~~ A to B

→ solving systems

$ax+by+cz=j$
$dx+ey+fz=k$
$gx+hy+iz=l$

$$\underset{[A]}{\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}} \cdot \underset{[B]}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{[C]}{\begin{bmatrix} j \\ k \\ l \end{bmatrix}}$$

you cannot divide matrices, but you <u>CAN</u> multiply by inverses

① multiply both sides by the inverse of A
② the left side should be A's multiplicative identity times matrix B and the right side should be $[A]^{-1} * [C]$
③ multiply like normal + find answers

7. Determinants of Larger Matrices

- must expand by minors : break big matrices into smaller matrices, solve for each smaller matrix, and combine
- do NOT forget about the checkerboard pattern (see III.5)

IV. Gauss Jordan Elimination

- a way to solve systems of equations
- goal is to go from $\left[\begin{array}{ccc|c} a & b & c & j \\ d & e & f & k \\ g & h & i & l \end{array}\right]$ to $\left[\begin{array}{ccc|c} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \end{array}\right]$
- use pivot rows and manipulate the matrix to get zeros in each correct place
  - ↳ replace numbers moving from left to right and top to bottom

V. Application of Matrices

① Finding the area of a triangle

- given 3 coordinate pairs, find area of triangle ΔABC

$$\left.\begin{array}{l} A\ (x_1, y_1) \\ B\ (x_2, y_2) \\ C\ (x_3, y_3) \end{array}\right\} \quad \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} = \frac{\text{determinant}}{2} = \text{area}$$

② Finding the equation of a line through two points

- given points (x_1, y_1) and (x_2, y_2)

$$\det \begin{bmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{bmatrix} = 0$$

- expand by minors
- solve for x and y

③ Finding the equation of a plane through 3 points

$$\left.\begin{array}{l} (x_1, y_1, z_1) \\ (x_2, y_2; z_2) \\ (x_3, y_3, z_3) \end{array}\right\} \det \begin{bmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{bmatrix} = 0$$

④ Cramer's Rule

can be used to find one single variable

① find determinant of matrix w/ constants (D)

② replace column (x, y, z) with answer column
↳ then find determinant (D_x or D_y or D_z)

③ divide new determinant with original determinant to solve

to find x:

$$\underbrace{\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix}}_{\text{find det (D)}} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \qquad D_x = \begin{bmatrix} a_1 & y_1 & z_1 \\ a_2 & y_2 & z_2 \\ a_3 & y_3 & z_3 \end{bmatrix} \qquad \boxed{X = \frac{D_x}{D}}$$

VI Secret Codes

→ <u>encoding matrix</u>: any square matrix with a determinant
↳ will be used to decode / encode message

→ <u>message matrix</u>: includes the secret message written in number form
↳ write message from top to bottom in column + left to right

this order: $\begin{bmatrix} 1 & 3 & 5 & 7 & 9 \\ 2 & 4 & 6 & 8 & 10 \end{bmatrix}$

→ <u>encoded message</u>: string of numbers that have no meaning until it is decoded

→ <u>decoding matrix</u>: the inverse of the encoding matrix

HOW to DECODE:

① multiply the decoding matrix by the encoded message to get the message matrix + then find message using key

How to encode:

① pick any encoding matrix

② write your message in message matrix form

③ multiply the encoding matrix by the message matrix to get encoded message (string of #s) and use key to decode.

VII. Transformations

$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
$(x,y) \to (x,-y)$
reflect over x

$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
$(x,y) \to (-x,y)$
reflect over y

$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
$(x,y) \to (-y,x)$
rotate 90° CCW

$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
$(x,y) \to (y,-x)$
rotate 270° CCW

$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$
$(x,y) \to (-x,-y)$
rotate 180°

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
$(x,y) \to (y,x)$
reflect over $y=x$

$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$
$(x,y) \to (-y,-x)$
reflect over $y=-x$

$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$
$(x,y) \to (kx,ky)$
dilation by factor of "k"

to dilate multiply scalar by the multiplicative identity to make transformational matrix

→ rotations:
matrix [T] rotates a figure CCW about the origin

$$[T] = \begin{bmatrix} \cos\theta & \cos(\theta+90) \\ \sin\theta & \sin(\theta+90) \end{bmatrix}$$

→ **Iterated Transformations**
preforming a transformation over and over again on the image of the previas transformation

VIII. Markov Chain Problems

- process that involves matrices in order to solve word problems
- make a probability matrix by dividing each entry by total possible out comes
  - ↳ the sum of each row should equal 1

Steady-state matrix:
↳ when the probabilities of a matrix do not change from one phase to the next after a long period of time
↳ won't change

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