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3RD

Accelerated Precalculus

3D Vectors Study Guide

Unit 12

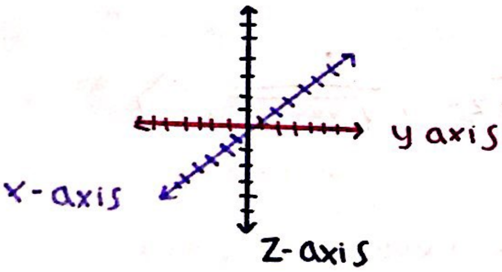
(I.) 2D Vectors

- Refer to 2D vectors study guide [unit 8]

(II.) Terminology & Notation

- **point** (x_1, y_1, z_1)
- **line** $\vec{r} = \langle x_1, y_1, z_1 \rangle + d\langle \Delta x, \Delta y, \Delta z \rangle$
- **plane** $Ax + By + Cz = D$
- **vector** $\langle x, y, z \rangle$ or $x\vec{i} + y\vec{j} + z\vec{k}$

(III). Drawing 3D Vectors



(IV.) 3D Vector Basics

→ writing 3D vector given 2 points

$A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$

$\boxed{\overrightarrow{AB} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle}$

→ adding / subtracting 3D vectors

- must add/subtract "matching parts" of each vector

$\overrightarrow{AB}\ \langle x_1, y_1, z_1 \rangle$ $\quad \overrightarrow{BC}\ \langle x_2, y_2, z_2 \rangle$

$\overrightarrow{AB} + \overrightarrow{BC} = \langle x_1 + x_2, y_1 + y_2, z_1 + z_2 \rangle$

→ midpoint of a line given 2pts

(x_1, y_1, z_1) and (x_2, y_2, z_2)

midpoint: $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$

→ noncollinear points

- points that do not lie on the same line

→ intercepts

1) x intercept: when y & z = 0
   $(x, 0, 0)$
2) y intercept: when x & z = 0
   $(0, y, 0)$
3) z intercept: when x & y = 0
   $(0, 0, z)$

→ magnitude of a 3D vector

$$|\langle x_1, y, z \rangle| = \boxed{\sqrt{x^2+y^2+z^2}}$$

→ unit vector of a 3D vector

- original vector written over the vector's magnitude

$\vec{v} = \langle x, y, z \rangle$

$$\vec{uv} = \left\langle \frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}} \right\rangle$$

V. Angle Between two vectors

$$\boxed{\theta = \cos^{-1}\left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}\right)}$$

- dot product divided by the product of the magnitudes

VI. Scalar Projections, Vector Projections & Orthogonal Components

→ scalar projection ($\vec{a}$ to $\vec{b}$)

$Sp = |\vec{a}| \cos\theta$ where θ = angle between vectors

$$\boxed{Sp = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}}$$

dot product divided by magnitude of 2nd vector

→ Vector Projections ($\vec{a}$ to $\vec{b}$)

vp = sp of $\vec{a}$ to $\vec{b}$ * unit vector $\vec{b}$

$$\boxed{vp = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2} * \vec{b}}$$

→ orthogonal component ($\vec{a}$ to $\vec{b}$)

oc = $\vec{a}$ − vp of ($\vec{a}$ to $\vec{b}$)

$$\boxed{oc = \vec{a} - \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2} * \vec{b}\right)}$$

VII Dot Product

• Represented by "•"

$\vec{A}\ \langle x_1\ \ y_1\ \ z_1 \rangle$

$\vec{B}\ \langle x_2\ \ y_2\ \ z_2 \rangle$

- the dot product of perpendicular vectors equals ZERO

$$\boxed{\vec{A} \cdot \vec{B} = x_1 x_2 + y_1 y_2 + z_1 z_2}$$

↳ multiply corresponding x, y, z terms & find the sum of the 3 values

VIII Cross Product

• aka vector product or outer product

• symbol "X"

→ the cross product of $\vec{r}$ and $\vec{s}$ ($\vec{r} \times \vec{s}$) is vector $\vec{t}$

1) $\vec{t}$ is perpindicular to both vectors

2) the magnitude of $\vec{t}$ is the area of the parallelogram formed by $\vec{r}$ and $\vec{s}$

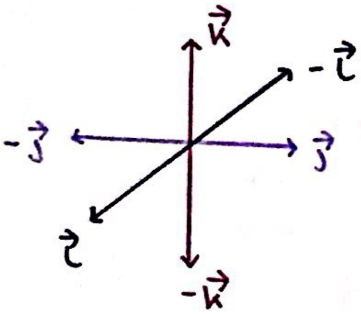
$\langle x_1, y_1, z_1 \rangle \times \langle x_2, y_2, z_2 \rangle$

$$\begin{vmatrix} i & j & k \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = \vec{i} \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} - \vec{j} \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} + \vec{k} \begin{vmatrix} x_1 & y_1 \\ x_2 & z_2 \end{vmatrix}$$

$$\boxed{(y_1 z_2 - y_2 z_1)\vec{i} - (x_1 z_2 - x_2 z_1)\vec{j} + (x_1 y_2 - x_2 y_1)\vec{k}}$$

Right Hand Rule

1) point hand in direction of 1st vector
2) curl fingers towards direction of 2nd vector
3) thumb will point in direction of the cross product



$$\begin{bmatrix} \vec{i} \times \vec{j} = \vec{k} \\ \vec{j} \times \vec{i} = -\vec{k} \end{bmatrix} \quad \begin{bmatrix} \vec{i} \times \vec{k} = -\vec{j} \\ \vec{k} \times \vec{i} = \vec{j} \end{bmatrix}$$

$$\begin{bmatrix} \vec{j} \times \vec{k} = \vec{i} \\ \vec{k} \times \vec{j} = -\vec{i} \end{bmatrix} \quad \begin{bmatrix} \vec{i} \times \vec{i} = 0 \\ \vec{j} \times \vec{j} = 0 \\ \vec{k} \times \vec{k} = 0 \end{bmatrix}$$

IX Application of Cross Products

→ If cross product is ⊥ to two vectors

- unit vector orthogonal to 2 vectors =

$$\vec{u} = \frac{\langle \text{cross product} \rangle}{\text{magnitude of cross product}}$$

→ If cross product = area ▱

- area of a triange ΔABC
  - $\vec{AB} \times \vec{AC}$ (cross product)
  - magnitude of cross product
  - divide by 2

X Distance between a point and a plane

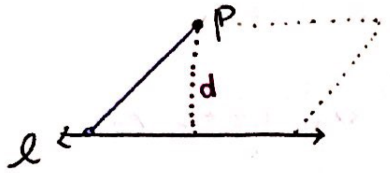
- point $P = (x, y, z)$
- plane $Ax + By + Cz = D$
  - ↳ $\langle A, B, C \rangle$ is a vector ⊥ to the plane

$$\boxed{d = \left| \frac{Ax + By + Cz - D}{\sqrt{A^2 + B^2 + C^2}} \right|}$$

Scanned with CamScanner

XI Distance between a point & a line

→ method 1: using area & cross products



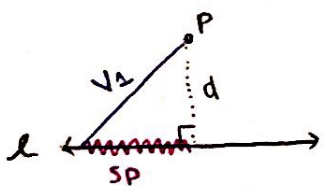
- make a vector between point P and any point on the line
- find the cross product of that vector with a vector that lies on the line (l)
- find the magnitude of the cross product ⇒ this is the area of the ▱ formed

area = bh

- area = magnitude of cross product
- base = magnitude of vector lying on line

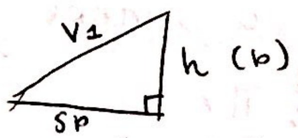
↳ solve for h (or the distance)

→ method 2: scalar projections & pythagorean theorem



- write a vector (v_1) between the point & any point on the line
- find the scalar projection of v_1 onto a vector lying on the line
- find the magnitude of v_1

pythag. theorem =



$a^2 + b^2 = c^2$

OR $(sp)^2 + b^2 = (|v_1|)^2$

• solve for b (height)

XII Distance between 2 skewed lines

$l_1 = \langle x_1\ y_1\ z_1 \rangle + d\langle A, B, C \rangle$

$l_2 = \langle x_2\ y_2\ z_2 \rangle + d\langle E, F, G \rangle$

1) find the vector between two points on the line

$\vec{V_1} = \langle x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1 \rangle$

2) find the cross product of the two vectors that lie on the lines

$$\begin{vmatrix} i & j & k \\ A & B & C \\ D & E & F \end{vmatrix} = \langle \quad , \quad , \quad \rangle = \vec{V_2}$$

3) find the <u>scalar projection</u> of V_1 onto V_2

(SP: vector between 2 points onto cross product of the lines)

XIII) Spheres

- in standard form: $\boxed{(x-x_1)^2 + (y-y_1)^2 + (z-z_1) = r^2}$
  - center at (x_1, y_1, z_1)
  - radius $= r$

* complete the square to get this form if given in general form

- intercepts: when the other two variables equal zero
- traces

$$\begin{bmatrix} xy \text{ trace} & \text{let } z=0 \\ xz \text{ trace} & \text{let } y=0 \\ yz \text{ trace} & \text{let } x=0 \end{bmatrix}$$

- surface area of a sphere

$$\boxed{SA = 4\pi r^2}$$

- volume of a sphere

$$\boxed{V = \frac{4}{3}\pi r^3}$$

XIV) Intersections

1) line and plane (0-1 points of intersection)
2) line and sphere (0-2 points of intersection)

- write parametric equation of the line
- substitute x, y, and z values into equation of the plane/sphere
- solve for T
- substitute T value(s) into parametric eq to find x, y, z to find P.O.I

XV Direction Cosines

- when the vector along the line is a unit vector, the coefficients are called direction cosines
  ↳ abbreviated c_1, c_2, c_3

  $\langle x, y, z \rangle$

$$c_1 = \frac{x}{\sqrt{x^2+y^2+z^2}} \qquad c_2 = \frac{y}{\sqrt{x^2+y^2+z^2}} \qquad c_3 = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

- use $\cos^{-1}$ to find direction angles (θ).

XVI Various Problem Solving

→ write the equation of a plane given 3 points. (A, B, C)

- write vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$
- find the cross product $\overrightarrow{AB} \times \overrightarrow{AC}$
- use cross products as the coefficients of x, y, and z in the plane
- plug in point A, B, or C into equation to solve for the constant (D)
  ↳ all 3 points should equal the same constant (D)

$$Ax + By + Cz = D$$

→ write a vector perpendicular to a plane

$Ax + By + Cz = D$ * use coefficients

↳ $\langle A, B, C \rangle$

→ write a line parallel to another line that passes through a given point

* works with two parallel planes & a point

- keep coefficients the same (slope = same)
- plug in given point
- solve for the new constant
- rewrite w/ variables and new constant

- write equation of a plane ⊥ to a line that passes through a point
  - write eq. of line
  - find plane ⊥ to the line
  - take coefficients, plug in the point, solve for the constant

- distance from a plane (or a line parallel to a plane) and another plane
  - find any point on the plane or line
  - find distance using distance formula

$$d = \left| \frac{Ax + By + Cz - D}{\sqrt{a^2+b^2+c^2}} \right|$$

- parallel ... planes to the axes/planes
  - plane parallel to the xy plane
    $z = k$ (constant)
  - plane parallel to the xz plane
    $y = k$ (constant)
  - plane parallel to the yz plane
    $z = k$ (constant)
  - line parallel to x axis
    ↳ perpendicular to yz plane
  - plane perpendicular to the y axis
    $y = k$ (constant)

- intersection between 2 planes
  - let 1 variable equal zero
  - use systems of equations to solve for the other two

↳ infinitely many solutions